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A q-deformed Schrödinger equation

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Abstract. We found Hermitian realizations of the position vector $\vec{r}$, the angular momentum $\vec{\Lambda}$ and the linear momentum $\vec{p}$, all behaving like vectors under the $su_q(2)$ algebra, generated by L_0 and $L_\pm$. They are used to introduce a q-deformed Schrödinger equation. Its solutions for the particular cases of the Coulomb and the harmonic oscillator potentials are given and briefly discussed.

1. Introduction

The general framework of the present study is the theory of quantum $su_q(2)$ algebra which has been the subject of extensive developments. Our purpose is to derive a q-deformed Schrödinger equation invariant under the $su_q(2)$ algebra. Here we discuss the case of spinless particles. So far, a general procedure (see, for example, [1–3]) was to write down the Hamiltonian in spherical coordinates and replace the $su(2)$ Casimir operator $C = \vec{L}^2$ by $C_q + f(q)$ where q is the deformation parameter, C_q the Casimir operator of the $su_q(2)$ algebra and $f(q)$ an arbitrary function with the property $f(q) \to 0$ when $q \to 1$. Of course this method introduces arbitrariness through the function f and sometimes anomalies as, for example, a bound spectrum [2] for the free Hamiltonian. Here we aim at removing such kind of arbitrariness and anomalies.

The novelty of our study is that by using the tensorial method we define Hermitian realizations of some elementary operators (position and angular momentum) behaving as *vectors* with respect to $su(2)_q$ algebra, generated by the operators L_0 and $L_\pm$. This allows us to introduce in a consistent way other operators, like the linear momentum and Hamiltonians having definite properties with respect to the deformed algebra. By this method the deformed parts of the operators are well defined, the only abitrary parts being the undeformed ones, as it will be further seen in the definition of the linear momentum. Proceeding on this line we shall also show that the angular momentum entering the expression of the Hamiltonian has components Λ_0 and $\Lambda_{\pm 1}$, different from L_0 and $L_\pm$. This leads to a proper behaviour of the free Hamiltonian. Here we consider two cases of central potentials: the harmonic oscillator and the Coulomb potential. Once the Hamiltonian is constructed we are able to derive both the spectrum and the eigenfunctions in a consistent way for each case. Our arguments are as follows.

The usual quantum mechanics of a point-like particle is constructed from two vectors: the position vector $\vec{r}$ and the linear momentum $\vec{p} = -\mathrm{i}\hbar\vec{\nabla}$. These two vectors are used to build

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all the other quantities as, for example, the angular momentum, the interaction potentials, etc, according to the classical rules. In general, these operators do not commute, their commutation relations following from the commutation relations of $\vec{r}$ and $\vec{p}$.

In a q-deformed quantum mechanics the commutation relations between the generators of the $su_q(2)$ algebra, L_i, and the position vector $\vec{r}$ are well defined inasmuch as $\vec{r}$ is considered a q-tensor of rank one (see section 2). Therefore, it is natural to take r_i as the basic quantities from which all the others should be built. Then, in deriving a q-deformed Schrödinger Hamiltonian, invariant under the $su_q(2)$ algebra, we searched for a realization of the linear momentum $\vec{p}$ entering the kinetic energy term. First it was necessary to find a realization for $\vec{r}$ and for L_i as *self-adjoint* operators obeying commutation relations characteristic to the deformed algebra. Then we looked for a realization of $\vec{p}$ in terms of $\vec{r}$ and of L_i. Observing that $\vec{p}$ can be written as a sum of two terms which are parallel and perpendicular to $\vec{r}$, respectively, we found that the deformation cannot be introduced in the radial part and conclude that this part must be independent of deformation. Here we take it to be just the undeformed one. The perpendicular part is explicitly deformed and writes as a vector product of $\vec{r}$ and of $\vec{\Lambda}$.

The paper is organized as follows. Section 2 contains the general commutation relations involving the q-angular momentum. We introduce some quantities having definite transformation properties with respect to the $su_q(2)$ algebra, namely the invariants C, C' and c and the vector $\vec{\Lambda}$ related to $\vec{L}$.

In section 3 we propose a realization of the position vector $\vec{r}$ and consistently of the q-angular momentum $\vec{L}$, in terms of spherical coordinates $r,\ x_0 = \cos\theta$ and φ, as for example in [4, 5].

The realization of the linear momentum $\vec{p}$ is considered in section 4. We first build the part of $\vec{p}$ perpendicular to $\vec{r}$, denoted by $\vec{\partial}$. This is achieved by using the cross product $\vec{r} \times \vec{\Lambda}$. We find that the components of $\vec{\partial}$ satisfy the same type of commutation relations as the components of $\vec{r}$.

Section 5 introduces the eigenfunctions of the q-deformed angular momentum written as power series of $x_0 = \cos\theta$. We show that the result is a generalization of the hypergeometric functions ${}_2F_1(a, b, c; \frac{1}{2}; x_0^2)$ and ${}_2F_1(a, b, c; \frac{3}{2}; x_0^2)$ which can be related to the q-deformed spherical functions $Y_{lm}(q, x_0, \varphi)$. Some useful properties and relations satisfied by the eigenfunctions are proved. In section 6, two particular cases of q-deformed Schrödinger equation containing a scalar potential are presented: the Coulomb and the three-dimensional harmonic oscillator. Their eigensolutions are given and the removal of the accidental degeneracy is discussed.

2. The q-angular momentum

The $su_q(2)$ algebra is generated by three operators $L_+,\ L_0$ and L_-, also named the q-angular momentum components. They have the following commutation relations:

$$[L_0, L_\pm] = \pm\, L_\pm \tag{1}$$

$$[L_+, L_-] = [2L_0] \tag{2}$$

where the quantity in square brackets is defined as

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{3}$$

In the following we shall introduce quantities having definite transformation properties with respect to the $su_q(2)$ algebra. They will further be used to build q-scalars and also q-vectors:

such as, for instance, the q-linear momentum entering the expression of the Hamiltonian operator.

First of all we recall that $su_q(2)$ algebra has an invariant C, called the Casimir operator

$$C = L_-L_+ + [L_0][L_0 + 1]. \tag{4}$$

Its eigenvalue associated to a $(2l + 1)$-dimensional irreducible representation is

$$C_l = [l][l + 1]. \tag{5}$$

By definition a q-vector in this algebra is given by a set of three quantities $v_k,\ k = 0, \pm 1$ satisfying the following relations:

$$[L_0, v_k] = kv_k \tag{6}$$

$$(L_\pm v_k - q^k v_k L_\pm)q^{L_0} = \sqrt{[2]}v_{k\pm1} \tag{7}$$

where $v_{\pm2}$ must be set equal to zero in the right-hand side of equation (7) when $k = \pm 1$. This definition is a particular case of an irreducible tensor of rank one (for the general case see, for example, [6]).

By comparing the relations (1), (2) with (6), (7) we observe that the operators L_k do not represent the components of a q-vector. Such an observation is also pointed out in [7] in the context of q-tensor operators for quantum groups. The situation is entirely different from the $su(2)$ algebra where L_k form a vector in the usual sense. However, one can use the components $L_\pm$ and L_0 to define a new vector $\vec{\Lambda}$ in the following manner:

$$\Lambda_{\pm1} = \mp 1\sqrt{\frac{1}{[2]}}q^{-L_0}L_\pm \tag{8}$$

$$\Lambda_0 = \frac{1}{[2]}(qL_+L_- - q^{-1}L_-L_+). \tag{9}$$

It is an easy matter to show that the operators Λ_k satisfy the relations (6) and (7). The vector $\vec{\Lambda}$ will be used in section 4 to construct the transverse part of the linear momentum $\vec{p}$.

Two q-vectors $\vec{u}$ and $\vec{v}$ satisfying equations (6) and (7) can be used to build a scalar S, according to the following definition:

$$S = \vec{u}\vec{v} = -\frac{1}{q}u_1v_{-1} + u_0v_0 - qu_{-1}v_1 \tag{10}$$

where the coefficients appearing in the sum are proportional to the q-Clebsch–Gordan coefficients $\langle 110|m - m0\rangle_q$. In this way the scalar product (10) becomes the ordinary scalar product of $R(3)$ when $q = 1$. By introducing a generalization of the cross product, two q-vectors can also be used to build another q-vector required by our approach, as will be shown in section 4.

In the case $\vec{u} = \vec{v} = \vec{\Lambda}$, the scalar product $\vec{\Lambda}^2$ defines a second invariant [8], C', which is not independent of C. The eigenvalue of C' is

$$C'_l = \frac{[2l]}{[2]}\frac{[2l + 2]}{[2]}. \tag{11}$$

Another invariant expression, c, defined as

$$c = q^{-2L_0} + \lambda\Lambda_0 \tag{12}$$

with

$$\lambda = q - \frac{1}{q} \tag{13}$$

will be frequently used in order to write the subsequent formulae in a more compact form. Its eigenvalue is

$$c_l = \frac{q^{2l+1} + q^{-2l-1}}{[2]}. \tag{14}$$

The invariants C, C' and c are not independent and can be written in terms of a single one. We list here some relations between their eigenvalues:

$$C'_l = \frac{2}{[2]} C_l + \frac{\lambda^2}{[2]^2} C_l^2$$
$$C_l = \frac{[2]}{\lambda^2}(c_l - 1)$$
$$C'_l = \frac{c_l + 1}{[2]} C_l.$$

It is worth noting that in the limit $q \to 1$ both C and C' turn into the Casimir invariant $C = \vec{L}^2$ of $su(2)$ with the eigenvalue $l(l+1)$, while c becomes equal to unity. The eigenvalues (11) and (14) will be used in section 4 to define the action of $\vec{p}^2$ on deformed spherical harmonics.

We recall that the action of the operators $L_\pm$ on the product $A_i B_j$ where A_i, B_j are monomials having the magnetic numbers i, j is written as

$$L_\pm(A_i B_j) = q^{-j}\, L_\pm(A_i) B_j + q^i A_i L_{\pm 1}(B_j). \tag{15}$$

Finally, we mention that the results listed in this section are valid for any realization of the $su_q(2)$ algebra.

3. The position vector $\vec{r}$ and a realization of L_0, $L_\pm$

In the $R_q(3)$ space we define the position vector $\vec{r}$ as having three noncommutative components r_1, r_0 and r_{-1}, satisfying the following relations:

$$\begin{aligned} r_0 r_{\pm 1} &= q^{\mp 2} r_{\pm 1} r_0 \\ r_1 r_{-1} &= r_{-1} r_1 + \lambda\, r_0^2. \end{aligned} \tag{16}$$

These equations are similar to equations (3.11) of [1]. They are typical for a noncommutative algebra and have been chosen in order to be compatible with the index raising and lowering operations. For instance, the results obtained by applying L_+ on $r_0 r_{+1}$ and on $r_{+1} r_0$ are compatible only if equation (16) holds.

The scalar quantity r^2 defined according to equation (10)

$$r^2 = \vec{r}^2 = -\frac{1}{q} r_1 r_{-1} + r_0^2 - q r_{-1} r_1 \tag{17}$$

commutes with all r_i and all L_i of equations (1) and (2), provided r_i $(i = 0, \pm 1)$ satisfy the conditions (6) and (7) to be a vector, which is the case here. For $q = 1$ the scalar r is nothing but the length of the position vector $\vec{r}$. We shall keep this meaning for $q \neq 1$ too.

Searching for concrete realizations of r_i, L_0 and $L_\pm$, we begin by expressing L_0 in spherical coordinates as in the $R(3)$ case:

$$L_0 = -\mathrm{i} \frac{\partial}{\partial \varphi}. \tag{18}$$

The next step is to write $\vec{r}$ as a product of r and of a unit vector $\vec{x}$, depending on angles, so we have

$$\begin{aligned} r_{\pm 1} &= r x_{\pm 1} \\ r_0 &= r x_0. \end{aligned} \tag{19}$$

It remains now to find a realization of $x_{\pm 1}$ in terms of the azimuthal angle φ and of x_0, which is in fact equal to $\cos\theta$, just as in the classical $R(3)$ case. From the relations (16), (17) and (19) one can find

$$\begin{aligned} x_1 x_{-1} &= -\frac{1}{[2]}(1-q^2 x_0^2) \\ x_{-1} x_1 &= -\frac{1}{[2]}(1-q^{-2} x_0^2). \end{aligned} \tag{20}$$

This suggests that equations (16) can be satisfied by simple forms of x_1 and x_{-1} provided a dilatation operator N_0 is introduced through the commutation relations

$$[N_0, x_0^n] = n x_0^n \tag{21}$$

and having the hermiticity property

$$N_0^+ = -N_0 - 1. \tag{22}$$

Then the realization of x_1 and x_{-1} satisfying (16) turns out to be

$$x_1 = -\mathrm{e}^{\mathrm{i}\varphi}\sqrt{\frac{q}{[2]}}\sqrt{1-q^2 x_0^2}\, q^{2N_0} \tag{23}$$

$$x_{-1} = \mathrm{e}^{-\mathrm{i}\varphi}\sqrt{\frac{1}{[2]q}}\sqrt{1-q^{-2} x_0^2}\, q^{-2N_0}. \tag{24}$$

The realizations (23), (24) will be discussed in more detail in appendix B.

Taking now into account relations (21) and (22) and assuming

$$x_0^+ = x_0 = \cos\theta \tag{25}$$

we get the expected hermiticity properties for $x_\pm$ as

$$x_1^+ = \frac{-1}{q} x_{-1} \tag{26}$$

$$x_{-1}^+ = -q x_1. \tag{27}$$

All these arguments allow us to conclude that equations (23)–(25) define the realization of the position vector $\vec{r}$ in the $R_q(3)$ space. (See appendix B.)

The following step is to search for a realization of the $su_q(2)$ generators. The expressions we propose for L_+ and L_- are

$$L_+ = \sqrt{[2]}\mathrm{e}^{\mathrm{i}\varphi}\tilde{x}_1^{L_0+1}\frac{1}{x_0}\frac{1-q^{-2N_0}}{1-q^{-2}}\tilde{x}_1^{-L_0} q^{L_0} \tag{28}$$

$$L_- = \sqrt{[2]}\mathrm{e}^{-\mathrm{i}\varphi}\tilde{x}_{-1}^{-L_0+1}\frac{1}{x_0}\frac{1-q^{2N_0}}{1-q^{2}}\tilde{x}_{-1}^{L_0} q^{L_0} \tag{29}$$

where $\tilde{x}_{\pm 1} = \mathrm{e}^{\mp\mathrm{i}\varphi} x_{\pm 1}$ depend on x_0 only. The reason why instead of $x_{\pm 1}$ we use here $\tilde{x}_{\pm 1}$, where the phase factor has been removed, is that expressions like $x_{\pm 1}^{L_0}$ have no meaning, while $\tilde{x}_{\pm 1}^{L_0}$ are well defined as discussed below equation (30). From equations (18), (28) and (29) we can now construct the Casimir operator C of equation (4). Its eigenfunctions are expected to be q-spherical functions as in [4]. For $q = 1$ they become ordinary spherical harmonics. Therefore, they can take the form

$$\tilde{Y}_{lm}(q, x_0, \phi) = \mathrm{e}^{\mathrm{i}m\varphi}\ \tilde{x}_1^m \Theta_{lm}(x_0) \tag{30}$$

where $\Theta_{lm}(x_0)$ are the q-analogue of the associated Legendre functions. The functions (30) will be derived and normalized in section 5.

The construction of the generators $L_\pm$ and their action on forms having definite m will be discussed in detail in appendix A. We demonstrate that expressions (18), (28) and (29) satisfy the commutation relations (1) and (2) and hence we conclude that they are the realization of the $su_q(2)$ generators in the $R_q(3)$ space.

In appendix B we show that $\vec{r}$, defined by (19)–(22), behaves indeed as a vector in this $su_q(2)$ algebra, since it satisfies the relations (6) and (7) with $\vec{L}$ given by (18), (28) and (29).

4. The q-linear momentum $\vec{p}$

The aim of this section is to introduce an expression of the linear momentum behaving like a q-vector. We try to follow the general line of undeformed quantum mechanics, that is, to define a position vector $\vec{r}$ and a linear momentum vector $\vec{p}$ which are the building blocks of the quantum mechanical formalism. In the undeformed space the derivative $\frac{\partial}{\partial \vec{r}}$ is a vector which is supposed to be proportional to the linear momentum. In a q-deformed quantum mechanics no such derivative exists and one is forced to search for other expressions.

In this approach we start by observing that the linear momentum can be written as made of two parts: a part perpendicular to $\vec{x}$ and another one parallel to it. The first one is defined with the aid of the cross product $\vec{x}\times\vec{\Lambda}$ and the second one is assumed to have the form $\vec{x}\,\frac{1}{r} f(r\frac{\partial}{\partial r}+1)$, where f is a function which will be defined in the following. Then the components of the transverse part, denoted by ∂_k, read

$$\partial_1 = q^{-1}x_1\Lambda_0 - qx_0\Lambda_1 + x_1 c \tag{31}$$

$$\partial_0 = x_1\Lambda_{-1} - \lambda x_0\Lambda_0 - x_{-1}\Lambda_1 + x_0 c \tag{32}$$

$$\partial_{-1} = -qx_{-1}\Lambda_0 + q^{-1}x_0\Lambda_{-1} + x_{-1}c \tag{33}$$

where c is the invariant defined in equation (12) and the terms $x_k\, c$ have been added to the cross product $\vec{x}\times\vec{\Lambda}$ in order to ensure the well-defined character with respect to the Hermitian conjugation operation

$$\partial_k^+ = -\left(-\frac{1}{q}\right)^k \partial_{-k}. \tag{34}$$

It can be checked that the quantities ∂_k form a vector as defined by equations (6) and (7). Moreover, they satisfy the following relations:

$$\partial_0\partial_1 = q^{-2}\partial_1\partial_0 \tag{35}$$

$$\partial_0\partial_{-1} = q^2\partial_{-1}\partial_0 \tag{36}$$

$$\partial_1\partial_{-1} = \partial_{-1}\partial_1 + \lambda\partial_0^2. \tag{37}$$

These equations are similar to (16) satisfied by the position vector. Equation (35) has been directly obtained by commuting ∂_0 with ∂_1. Equation (36) is the Hermitian conjugate of the above one. Equation (37) can be obtained either from (35) or (36) by using the relation (7).

Also, by multiplying equations (31)–(33) with the corresponding x_k and taking into account the commutation relations (6) and (7) one gets

$$\vec{x}\vec{\partial} = -\vec{\partial}\vec{x} = c. \tag{38}$$

By commuting the invariant c with $\vec{x}$ one finds

$$\vec{\partial} = \lambda^{-2}[c, \vec{x}]. \tag{39}$$

Taking now the matrix elements of the last relation one obtains

$$\langle l+1 m'|\vec{\partial}|lm\rangle = \frac{[2l+2]}{[2]}\langle l+1 m'|\vec{x}|lm\rangle \tag{40}$$

$$\langle l-1\; m'|\vec{\partial}|lm\rangle = -\frac{[2l]}{[2]}\langle l-1\; m'|\vec{x}|lm\rangle. \tag{41}$$

From parity arguments one can also write

$$\langle lm'|\partial_k|lm\rangle = 0. \tag{42}$$

The matrix elements of $\vec{x}$ can be calculated (see the next section) so that from replacing the matrix elements of $\vec{\partial}$ by those of $\vec{x}$ with the aid of equations (40) and (41) one can obtain the eigenvalues of $\vec{\partial}^2$. These are

$$\langle lm|\vec{\partial}^2|lm\rangle = -\frac{[2l]}{[2]}\frac{[2l+1]}{[2]} - c_l^2. \tag{43}$$

At the beginning of this section we mentioned that the component of $\vec{p}$ parallel to $\vec{x}$ is assumed to have the form $\vec{x}\frac{1}{r}f(r\frac{\partial}{\partial r}+1)$. For simplicity we take here $f(x) = x$. In this case the realization of the q-linear momentum $\vec{p}$ is

$$\vec{p} = \frac{-\mathrm{i}}{r}\left(\vec{x}(r\frac{\partial}{\partial r}+1) - \vec{\partial}\right). \tag{44}$$

Then using equations (38) and (43) one can write

$$\vec{p}^2\tilde{Y}_{lm} = \left[-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}+1\right) + \frac{1}{r^2}\left(\frac{[2l]}{[2]}\frac{[2l+2]}{[2]} + c_l^2 - c_l\right)\right]\tilde{Y}_{lm}. \tag{45}$$

One can see that in the limit $q \to 1$ one recovers the action of the Laplace operator on a spherical harmonic which justifies our choice for f.

We mention that it is a simple but tedious matter to calculate the commutation relations between $\vec{r}$ and $\vec{p}$ and to verify that one gets the right result for $q = 1$. We do not display these commutation relations here because they are rather intricate and unnecessary in the derivation of a covariant Schrödinger equation.

We also note that the operator $\vec{\Lambda}$, behaving as a vector under the $su_q(2)$ algebra, can be written as a cross product of $\vec{r}$ and $\vec{p}$, but this does not bring any simplification because of the commutation relations between $\vec{r}$ and $\vec{p}$.

5. The eigenfunctions of the q-angular momentum

By definition, the basis vectors $\Phi_{lm}(q, x_0, \varphi)$ forming an invariant subspace for a $(2l+1)$-dimensional irreducible representation of $su_q(2)$ are eigenfunctions of L_0 and of the Casimir operator C of equation (4). We begin by writing them as polynomials in x_0 multiplied by x_1^m:

$$\Phi_{lm}(q, x_0, \varphi) = x_1^m \sum_{k\geqslant 0} a_k x_0^k \tag{46}$$

where the sum runs either over k even when $l-m$ is even or over k odd when $l-m$ is odd. In both cases it runs up to $l-m$ but it starts at zero for $(l-m)$ even and at one for $(l-m)$ odd.

As for the $R(3)$ case, the basic equation which determines the matrix elements of L_+ and L_- is

$$L_+L_-\Phi_{lm}(q, x_0, \varphi) = [l+m][l-m+1]\Phi_{lm}(q, x_0, \varphi). \tag{47}$$

This equation leads to the recursion relation

$$a_{k+2} = -q^{-2m}\frac{[l-m-k][l+m+k+1]}{[k+1][k+2]}a_k. \tag{48}$$

Then taking $a_0 = 1$ we obtain for $(l-m)$ even

$$\Phi_{lm}(q, x_0, \varphi) = x_1^m\left\{1 - \frac{[l-m][l+m+1]}{[2]!}(q^{-m}x_0)^2 + \frac{[l-m][l-m-2][l+m+1][l+m+3]}{[4]!}(q^{-m}x_0)^4 - \cdots\right\} \tag{49}$$

while for $(l-m)$ odd we get

$$\Phi_{lm}(q,x_0,\varphi) = x_1^m \left\{ \frac{1}{[1]!}(q^{-m}x_0) - \frac{[l-m-1][l+m+2]}{[3]!}(q^{-m}x_0)^3 + \frac{[l-m-1][l-m-3][l+m+2][l+m+4]}{[5]!}(q^{-m}x_0)^5 - \cdots \right\}. \tag{50}$$

In order to express these results in terms of a q-hypergeometric series it is necessary to write all the q-numbers $[n]$ in the form

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = [2]\frac{(q^2)^{\frac{n}{2}} - (q^2)^{-\frac{n}{2}}}{q^2 - q^{-2}} = [2]\left[\frac{n}{2}\right]_{q^2}. \tag{51}$$

For $(l-m)$ even we have then

$$\Phi_{lm}(q,x_0,\varphi) = x_1^m \, {}_2F_1\left(q^2; \frac{l+m+1}{2}, \frac{-l+m}{2}; \frac{1}{2}; q^{-m}x_0^2\right) \tag{52}$$

while for $(l-m)$ odd we get

$$\Phi_{lm}(q,x_0,\varphi) = x_1^m q^{-m} x_0 \, {}_2F_1\left(q^2; \frac{l+m+2}{2}, \frac{-l+m+1}{2}; \frac{3}{2}; (q^{-m}x_0)^2\right). \tag{53}$$

The argument q^2 in ${}_2F_1$ specifies that all the q-numbers in the series expansion of ${}_2F_1$ must be calculated with q^2 instead of q.

Moreover, we found that the functions $\Phi_{lm}(q,x_0,\varphi)$ satisfy the following simple relations:

$$x_1 \frac{1}{x_0} \frac{1-q^{-2N_0}}{1-q^{-2}} \Phi_{lm}(q,x_0,\varphi) = -[l-m][l+m+1]\Phi_{lm+1}(q,x_0,\varphi) \tag{54}$$

for $(l-m)$ even, and

$$x_1 \frac{1}{x_0} \frac{1-q^{-2N_0}}{1-q^{-2}} \Phi_{lm}(q,x_0,\varphi) = \Phi_{lm+1}(q,x_0,\varphi) \tag{55}$$

for $(l-m)$ odd.

The normalized eigenfunctions of C and L_0 take now the form

$$Y_{lm}(q,x_0,\varphi) = (-1)^{\frac{l-m}{2}} \sqrt{\frac{[2l+1]}{4\pi}} \left(\frac{[l-m-1]!!}{[l-m]!!} \frac{[l+m-1]!!}{[l+m]!!} \right)^{1/2} [2]^{\frac{m}{2}} \Phi_{lm}(q,x_0,\varphi) \tag{56}$$

for $(l-m)$ even, and

$$Y_{lm}(q,x_0,\varphi) = (-1)^{\frac{l-m-1}{2}} \sqrt{\frac{[2l+1]}{4\pi}} \left(\frac{[l-m]!!}{[l-m-1]!!} \frac{[l+m]!!}{[l+m-1]!!} \right)^{1/2} [2]^{\frac{m}{2}} \Phi_{lm}(q,x_0,\varphi) \tag{57}$$

for $(l-m)$ odd. Their orthogonality relation becomes

$$\int Y^+_{l'm'}(q,x_0,\varphi) Y_{lm}(q,x_0,\varphi) \, \mathrm{d}\varphi \, \mathrm{d}[x_0] = \delta_{ll'}\delta_{mm'} \tag{58}$$

where the integral over φ is the same as for spherical harmonics, while the integral over $\mathrm{d}[x_0]$ defined on the interval $(-1, 1)$ is the sum of

$$\int_0^1 x_0^n \, \mathrm{d}[x_0] = \frac{1}{[n+1]} \tag{59}$$

and

$$\int_{-1}^0 x_0^n \, \mathrm{d}[x_0] = (-1)^n \frac{1}{[n+1]}. \tag{60}$$

The phase appearing in the right-hand side of the integral (60) is due to parity arguments. Relation (59) is, in fact, the result of a discrete integration of $f(x_0) = x_0^n$, performed by dividing the integration interval $(0, 1)$ in an infinite set of segments located between two succesive points $x_k = q^k$ and $x_k = q^{k+1}$ where $q < 1$

$$\int_0^1 f(x_0)\,\mathrm{d}[x_0] = \sum_{k=0}^{\infty} f(x_{2k+1})(x_{2k} - x_{2k+2}). \tag{61}$$

Looking now for the properties of Y_{lm}, just as in the $R(3)$ case, we found that the product $x_k\ Y_{lm}$ can be expressed in terms of $Y_{l+1,\ m+k}$ or $Y_{l-1,\ m+k}$ as follows:

$$\begin{aligned} x_1 Y_{lm}(q, x_0, \varphi) = {} & q^{l-m}\sqrt{\frac{[l+m+1][l+m+2]}{[2][2l+1][2l+3]}}Y_{l+1m+1}(q, x_0, \varphi) \\ & -q^{-l-m-1}\sqrt{\frac{[l-m][l-m-1]}{[2][2l+1][2l-1]}}Y_{l-1m+1}(q, x_0, \varphi) \end{aligned} \tag{62}$$

$$\begin{aligned} x_0 Y_{lm} = {} & q^{-m}\sqrt{\frac{[l-m+1][l+m+1]}{[2l+1][2l+3]}}Y_{l+1m}(q, x_0, \varphi) \\ & -q^{-m}\sqrt{\frac{[l-m][l+m]}{[2l+1][2l-1]}}Y_{l-1\ m}(q, x_0, \varphi) \end{aligned} \tag{63}$$

$$\begin{aligned} x_{-1} Y_{lm}(q, x_0, \varphi) = {} & q^{l-m}\sqrt{\frac{[l-m+1][l-m+2]}{[2][2l+1][2l+3]}}Y_{l+1m-1}(q, x_0, \varphi) \\ & -q^{l-m+1}\sqrt{\frac{[l+m][l+m-1]}{[2][2l+1][2l-1]}}Y_{l-1m-1}(q, x_0, \varphi). \end{aligned} \tag{64}$$

In addition, we have found three relations which express the noncommutativity of x_k with Y_{lm} and represent a generalization of the equations (16):

$$x_0 Y_{lm}(q, x_0, \varphi) = q^{-2m}\ Y_{lm}(q, x_0, \varphi) x_0 \tag{65}$$

$$\begin{aligned} x_1 Y_{lm}(q, x_0, \varphi) = {} & Y_{lm}(q, x_0, \varphi) x_1 \\ & + \frac{\lambda}{\sqrt{[2]}} q^{-m-1}\sqrt{[l-m][l+m+1]}Y_{lm+1}(q, x_0, \varphi) x_0 \end{aligned} \tag{66}$$

$$\begin{aligned} x_{-1} Y_{lm}(q, x_0, \varphi) = {} & Y_{lm}(q, x_0, \varphi) x_{-1} \\ & - \frac{\lambda}{\sqrt{[2]}} q^{-m+1}\sqrt{[l+m][l-m+1]}Y_{lm-1}(q, x_0, \varphi) x_0. \end{aligned} \tag{67}$$

The last two equations have been obtained from (65) by acting with L_+ or L_- which leads to a rising or lowering of m in Y_{lm}.

6. A q-deformed Schrödinger equation

Taking into account all the above results, we assume that the Hamiltonian entering the q-deformed Schrödinger equation is

$$\mathcal{H} = \tfrac{1}{2}\vec{p}^2 + V(r) \tag{68}$$

where operator $\vec{p}$ has been defined in section 4. The eigenfunctions of this Hamiltonian are

$$\Psi(r, x_0, \varphi) = r^L u_L(r) Y_{lm}(q, x_0, \varphi) \tag{69}$$

where L is the solution of the following equation:

$$L(L+1) = \frac{[2l]}{[2]}\frac{[2l+2]}{[2]} + c_l^2 - c_l \qquad (70)$$

obtained from the requirement that $u_L(r)$ remains finite in the limit $r \to 0$.

This Schrödinger equation has simple solutions for the Coulomb potential $V(r) = -r^{-1}$ and for the oscillator potential $V(r) = \frac{1}{2}r^2$. The eigenvalues of the two Hamiltonians are

$$(E_{nl})_{Coulomb} = -\frac{1}{2(n+L+1)^2} \qquad (71)$$

for the Coulomb potential and

$$(E_{nl})_{oscillator} = (2n + L + \tfrac{3}{2}) \qquad (72)$$

for the oscillator potential, n being the radial quantum number and L the solution of equation (70), usually not an integer. We notice that the spectrum is degenerate with respect to the magnetic quantum number m, i.e. the essential degeneracy subsists. But the eigenvalues (71) and (72) depend on two quantum numbers so that the accidental degeneracy of the $q = 1$ case is removed. The dependence of eigenvalues on q can be obtained through solving equation (70) for L.

The solution of the wave equation which does not depend on θ and φ gives for the expectation value of x_0^2 the value $R^2/[3]$ instead of $R^2/3$ obtained in the case of spherical symmetry. The quantity R^2 denotes the expectation value of the operator r^2 in each case. It then results that the quadrupole moment as well as all the 2^{2n}-poles are different from zero, although the wavefunction does not depend on θ and φ. This clearly shows that the Hamiltonian (68)–(70) has lost the spherical symmetry. One can mention, however, that it gained another one, namely the symmetry under the $su_q(2)$ algebra which may have new physical implications.

We remark that there are three sources producing differences in the eigenvalue problem between the case of q-deformed Schrödinger equation and the case of spherical symmetry. The first one is that the q-functions $Y_{lm}(q, x_0, \varphi)$ differ from the spherical harmonics $Y_{lm}(\theta, \varphi)$ as shown in section 5. The second reason is that the coefficient of the centrifugal potential in the radial Schrödinger equation is proportional to $L(L+1)$, with L given by equation (70), and not to $l(l+1)$, as in the sperical case. The third source is that in the q-deformed case the integral over x_0 is performed according to the relations (58)–(60).

As a final comment let us recall that for $l = 0$ one has $c_l = 1$, hence $L = 0$. As a consequence the $l = 0$ levels are independent of the deformation parameter both for the harmonic oscillator and the Coulomb potential. An important physical aspect is that the centrifugal barrier disappears for $l = 0$ in contrast to the Hamiltonian H_q of [2]. Moreover, the whole Coulomb spectrum of [2] is different from ours. A careful analysis of the results shows that the differences come from the different forms of the centrifugal terms in the Hamiltonian.

Physical applications with numerical examples of the q-deformed Coulomb and harmonic oscillator spectra will be considered elsewhere.

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Appendix A.

We explain here in detail the construction (28), (29) of the operators $L_\pm$.

We start by observing that, according to the realization (23)–(25) the most general expression having the positive magnetic number m can be written as a series:

$$\Psi_m = \sum_k a_k \Psi_{mk} = x_1^m \sum_k a_k x_0^k. \tag{A.1}$$

We now raise the magnetic number m to $m+1$ of a single term $\Psi_{mk} = x_1^m\, x_0^k$ in the series by acting on it with the operator L_+ according to the rule (15)

$$L_+\Psi_{mk} = L_+ x_1^m x_0^k = \sqrt{[2]} x_1^{m+1} x_0^{k-1} \frac{1-q^{-2k}}{1-q^{-2}} q^m. \tag{A.2}$$

The result (A.2) shows that a possible realization of L_+ is

$$L_+ = \sqrt{[2]} x_1 \frac{1}{x_0} \frac{1-q^{-2N_0}}{1-q^{-2}} q^{L_0} \tag{A.3}$$

provided one forbids the derivative operator contained in $x_1\, \frac{1}{x_0}\, \frac{1-q^{-2N_0}}{1-q^{-2}}$ to act on x_1 which is a function of x_0. (See definition (23) of x_1.) This can be achieved by performing the following operations on the expression (A.3) of L_+: one multiplies it to the right with $x_1^{-L_0}$ in order to remove x_1^m from $\Psi_m^{(k)}$ and to the left with $x_1^{L_0+1}$ in order to create the factor x_1^{m+1} entering the expression of $\Psi_{m+1}^{(k-1)}$. The generator L_+ then reads as

$$L_+ = \sqrt{[2]} e^{i\varphi} \tilde{x}_1^{L_0+1} \frac{1}{x_0} \frac{1-q^{-2N_0}}{1-q^{-2}} \tilde{x}_1^{-L_0} q^{L_0} \tag{A.4}$$

where $\tilde{x}_1$ defined by

$$x_1 = e^{i\varphi} \tilde{x}_1 \tag{A.5}$$

has been introduced in order to have a clear notation for powers like $x_{\pm 1}^{L_0}$.

A similar problem occurs in quantum mechanics, but there the procedure eliminating the derivation of x_1 is different. The clue is that the result of the derivative $\frac{\partial}{\partial\theta} x_1^m$ is exactly cancelled out by the term i ctg$\theta\, \frac{\partial}{\partial\varphi}$ in the expression of the generator.

By taking now the Hermitian conjugate of (A.4) we obtain the realization of the generator L_-

$$L_- = \sqrt{[2]} e^{-i\varphi} \tilde{x}_{-1}^{-L_0+1} \frac{1}{x_0} \frac{1-q^{2N_0}}{1-q^{2}} \tilde{x}_{-1}^{L_0} q^{L_0}. \tag{A.6}$$

It remains only to show that L_0 defined in (18) and $L_\pm$ defined above satisfy the commutation relations (1) and (2). It is easy to see that the commutation relations (1) are satisfied if L_0 has the expression (18). In order to prove the relation (2) we write separately the two terms of the commutator:

$$L_+L_- = [2] \tilde{x}_1^{L_0} \frac{1}{x_0} \frac{1-q^{-2N_0}}{1-q^{2}} \tilde{x}_1^{-L_0+1} \tilde{x}_{-1}^{-L_0+1} \frac{1}{x_0} \frac{1-q^{2N_0}}{1-q^{2}} \tilde{x}_{-1}^{L_0} q^{2L_0-1} \tag{A.7}$$

$$L_-L_+ = [2] \tilde{x}_{-1}^{L_0} \frac{1}{x_0} \frac{1-q^{2N_0}}{1-q^{2}} \tilde{x}_{-1}^{L_0+1} \tilde{x}_1^{L_0+1} \frac{1}{x_0} \frac{1-q^{-2N_0}}{1-q^{-2}} \tilde{x}_1^{-L_0} q^{2L_0-1}. \tag{A.8}$$

We notice then that the factor $\tilde{x}_1^{-L_0+1} \tilde{x}_{-1}^{-L_0+1}$ in (A.7) can be written as a product of $(-L_0+1)$ parentheses:

$$\tilde{x}_1^{-L_0+1} \tilde{x}_{-1}^{-L_0+1} = \left(-\frac{1}{[2]}\right)^{-L_0+1} (1-q^2x_0^2)(1-q^6x_0^2)\ldots((1-q^{-4L_0+2}x_0^2) \tag{A.9}$$

and hence it can be replaced by $\tilde{x}_1^{-L_0}\ \tilde{x}_{-1}^{-L_0}$ when moving over $\frac{1}{x_0}\ \frac{1-q^{2N_0}}{1-q^2}$ to the right, up to the place in front of $\tilde{x}_{-1}^{L_0}$. One obtains in this way an expression having the factor $\tilde{x}_1^{-L_0}$ in the extreme right and $\tilde{x}_1^{L_0}$ in the extreme left. An analogous transformation can be performed in equation (A.8)) by moving $\tilde{x}_{-1}^{L_0+1}\tilde{x}_1^{L_0+1}$ and getting the same factors in the extreme right and left. The difference of the two equations is then

$$L_+L_- - L_-L_+ = \tilde{x}_1^{L_0}[2L_0]\tilde{x}_1^{-L_0}. \tag{A.10}$$

Moving now $\tilde{x}_1^{L_0}$ to the right in order to cancel $\tilde{x}_1^{-L_0}$ we get

$$L_+L_- - L_-L_+ = [2L_0] \tag{A.11}$$

as required.

Finally, considering the limit $q \to 1$ we notice that the expression $\frac{1}{x_0}\frac{1-q^{2N_0}}{1-q^2}$ goes to $\frac{\partial}{\partial x_0}$ and the realization (A3) of L_+ becomes

$$L_+(q=1) = \sqrt{2}\mathrm{e}^{\mathrm{i}\varphi}\tilde{x}_1^{L_0+1}\frac{\partial}{\partial x_0}\tilde{x}_1^{-L_0} \tag{A.12}$$

where $\tilde{x}_1$ is now

$$\tilde{x}_1 = -\frac{1}{\sqrt{2}}\sqrt{1-x_0^2} = -\frac{\sin\theta}{\sqrt{2}}. \tag{A.13}$$

Performing the derivative with respect to x_0 in (A.12) we get, after introducing the spherical coordinates,

$$L_+(q=1) = -\sqrt{2}\mathrm{e}^{\mathrm{i}\varphi}\left(\frac{\sin\theta}{-\sqrt{2}}\right)^{L_0+1}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\left(\frac{\sin\theta}{-\sqrt{2}}\right)^{L_0} = \mathrm{e}^{\mathrm{i}\varphi}\left(\frac{\partial}{\partial\theta} - L_0\,\mathrm{ctg}\,\theta\right) \tag{A.14}$$

which is the classical expression.

Appendix B.

In this appendix we show that $x_{\pm1,0}$ defined by (23)–(25) satisfy the relations (6), (7) provided x_i satisfy the commutation relations (16) and $L_\pm$ are given by equations (28), (29).

First we write explicitly the commutation relation (16)

$$\begin{aligned}(L_+x_1 - qx_1L_+)q^{L_0} &= \sqrt{[2]}\mathrm{e}^{\mathrm{i}\varphi}\tilde{x}_1^{L_0+1}\frac{1}{x_0}\frac{1-q^{-2N_0}}{1-q^{-2}}\tilde{x}_1^{-L_0}q^{L_0}\tilde{x}_1\mathrm{e}^{\mathrm{i}\varphi}\\ &\quad -q\tilde{x}_1\mathrm{e}^{\mathrm{i}\varphi}\mathrm{e}^{\mathrm{i}\varphi}\tilde{x}_1^{L_0+1}\frac{1}{x_0}\frac{1-q^{-2N_0}}{1-q^{-2}}\tilde{x}_1^{-L_0}q^{L_0}\tilde{x}_1\end{aligned} \tag{B.1}$$

and demonstrate that x_1 is the highest component. Indeed, by moving $\mathrm{e}^{\mathrm{i}\varphi}$ from right to left in the first term in the right-hand side of the above equation we get

$$\sqrt{[2]}\mathrm{e}^{2\mathrm{i}\varphi}\left(\tilde{x}_1^{L_0+2}\frac{1}{x_0}\frac{1-q^{-2N_0}}{1-q^{-2}}\tilde{x}_1^{-L_0-1}q^{L_0+1}\tilde{x}_1 - q\tilde{x}_1^{L_0+2}\frac{1}{x_0}\frac{1-q^{-2N_0}}{1-q^{-2}}\tilde{x}_1^{-L_0}q^{L_0}\right) = 0. \tag{B.2}$$

Next, by acting with L_- on x_1 we get

$$\begin{aligned}(L_-x_1 - qx_1L_-)\,q^{L_0} &= \sqrt{[2]}\left(\mathrm{e}^{-\mathrm{i}\varphi}\tilde{x}_{-1}^{-L_0+1}\frac{1}{x_0}\frac{1-q^{2N_0}}{1-q^2}\tilde{x}_{-1}^{L_0}q^{L_0}\tilde{x}_1\mathrm{e}^{\mathrm{i}\varphi}\right.\\ &\quad \left.-q\mathrm{e}^{\mathrm{i}\varphi}\tilde{x}_1\mathrm{e}^{-\mathrm{i}\varphi}\tilde{x}_{-1}^{-L_0+1}\frac{1}{x_0}\frac{1-q^{2N_0}}{1-q^2}\tilde{x}_{-1}^{L_0}q^{L_0}\right)q^{L_0+1}\\ &= \sqrt{[2]}\left(\tilde{x}_{-1}^{-L_0}\frac{1}{x_0}\frac{1-q^{2N_0}}{1-q^2}\tilde{x}_{-1}^{L_0+1}\tilde{x}_1 - \tilde{x}_1\tilde{x}_{-1}^{-L_0+1}\frac{1}{x_0}\frac{1-q^{2N_0}}{1-q^2}\tilde{x}_{-1}^{L_0}\right)q^{L_0+1}.\end{aligned} \tag{B.3}$$

Now, using equations (20) we have for the right-hand side of (B.3):

$$\left(\tilde{x}_{-1}^{-L_0}\frac{1}{x_0}\frac{1-q^{2N_0}}{1-q^2}\tilde{x}_{-1}^{L_0}\frac{1-q^{-2}x_0^2}{-[2]}-\frac{1-q^2\,x_0^2}{-[2]}\tilde{x}_{-1}^{-L_0}\frac{1}{x_0}\frac{1-q^{2N_0}}{1-q^2}\tilde{x}_{-1}^{L_0}\right)q^{L_0+1}=\sqrt{[2]}x_0 \quad \text{(B.4)}$$

as expected.

The last commutation relation we give explicitly is

$$(L_-x_0-x_0L_-)\,q^{L_0}=\sqrt{[2]}\left(\mathrm{e}^{-\mathrm{i}\varphi}\tilde{x}_{-1}^{-L_0}\frac{1}{x_0}\frac{1-q^{2N_0}}{1-q^2}\tilde{x}_{-1}^{L_0}q^{L_0}x_0\right.$$
$$\left.-x_0\mathrm{e}^{-\mathrm{i}\varphi}\tilde{x}_{-1}^{-L_0}\frac{1}{x_0}\frac{1-q^{2N_0}}{1-q^2}\tilde{x}_{-1}^{L_0}\tilde{x}_{-1}^{L_0}q^{L_0}\right)q^{L_0}. \quad \text{(B.5)}$$

Proceeding as above and moving x_0 up to $\frac{1}{x_0}$ it can be easily shown that the right-hand side of (B.5) is actually equal to $\sqrt{[2]}\,x_{-1}$.

In a similar manner it can be proved that x_{-1} is the lowest component (see relations (B.1), (B.2)) and that acting with L_+ on x_{-1} and x_0 one gets x_0 and x_{+1}, respectively. This completes our proof.

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